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Compact Topologies on Minkowski Space

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Abstract

Minkowski space can be given topologies which are compact and which have, as their homeomorphism group, the inhomogeneous Lorentz group together with dilatations.

1. *Introduction*

Several years ago, Everett & Ulam (1948) asked the following question: Given a set X and a group G, which topologies on X have G as their homeomorphism group? Apart from the mathematical interest, the solution to this question will be interesting in several other respects, especially in the case of some known spaces and some well-known groups acting on them. For example, if one is dealing with a concrete set such as the four-dimensional space-time continuum of Special Relativity and a group such as the inhomogeneous Lorentz group, then one would like to know the different topologies on the set which would give rise to the same homeomorphism group, namely the inhomogeneous Lorentz group. However, it has been observed in this case (Nanda, 1971, 1972; Zeeman, 1967) that there exist several such topologies, some of which are not even comparable to each other. It is therefore obvious that the solution to such a problem will be immensely difficult and in some cases a highly improbable task. As a first step, therefore, it will be desirable to start with a given topology and to look for another topology on the same set which is, in some way, connected with the given topology and whose homeomorphism group will be the same. Lee (1967, 1969) has already done some work in this direction. The object of the present paper is to give one such topology.

We shall start with an arbitrary topological space T and its antispace T^* (definitions given in the next section). We shal prove that the group of homeomorphisms of T is equal to that of its antispace T^*. De Groot (1967) has shown that if a topological space T is first countable and Hausdorff but not compact, then its antispace T^* will be compact, non-Hausdorff, T_1, and that every open set in T^* will be connected. Furthermore, the topology of T^* will be weaker than that of T. At the end of this paper we give examples of some first countable Hausdorff topologies on Minkowski space (definitions given in Section 4) and determine the corresponding compact topologies of their antispaces. We shall also show that for any Hausdorff topology on Minkowski space having the inhomogeneous Lorentz group as the homeomorphism group, the open sets are unbounded in the Euclidean sense and are therefore of infinite diameter.

2. *Preliminaries*

The following definitions are due to De Groot (1967).

Definitions

Let X be any set and let $\{G\}$ be a family of subsets G of X closed under finite unions and arbitrary intersections. We do not assume the usual convention that X and the empty set $\emptyset$ are members of $\{G\}$. A pair $T_- = (X, \{G\})$ is called a (topological) *Minus space*, where $\{G\}$ indicates the family of closed sets of T_-. One can of course extend every minus space T_- to a topological space T by adding X and $\emptyset$ as closed sets. Let $T_- = (X, \{G\}, \{C\})$ and $T^*_- = (X, \{C\}, \{G\})$ be two minus spaces over X where $\{G\}$ denotes the family of all closed sets G and $\{C\}$ denotes the family of all compact sets C in T_-, while $\{C\}$ are the closed sets of T^*_- and $\{G\}$ are the compact sets of T^*_-. So the identity map of X onto itself maps the closed (compact) sets of T_- onto the compact (closed) sets of T^*_-. Such a pair, T_-, T^*_-, is called an *antipair* and they are called the *antispaces* of each other. A space T_- is called an antispace if there exists a T^*_- as indicated above.

For more detailed properties and results about minus spaces and antispaces, see De Groot (1967).

We shall now prove the following:

3. *The Main Theorem*

Theorem 1. The group of homeomorphisms of a topological space T is the same as that of its antispace T^*.

Proof. Let X be any nonemtpy set and $T = (X, \{G\}, \{C\})$, $T^* = (X, \{C\}, \{G\})$ be two antispaces of each other over X where $\{G\}$ denotes the family of all closed sets G and $\{C\}$ denotes the family of all compact sets C of T while $\{C\}$ are the closed sets and $\{G\}$ are the compact sets of T^*. Let f belong to the homeomorphism group of T and C be any closed set of T^*, then C is compact in T. Since the continuous image of a compact set is compact, $f(C)$ is compact in T. Consequently, $f(C)$ belongs to $\{C\}$ in T^*. Thus, f maps any closed set in T^* to a closed set in T^*. Similarly, we can prove that f^{-1} maps any closed set in T^* to a closed set in T^*. Moreover, f as a map over the set X is one to one and onto and therefore is a homeomorphism of T^*. Conversely, it can easily be proved that if f is a homeomorphism of T^*, then it is also a homeomorphism of T. This completes the proof of the theorem.

4. *Examples*

It has been shown by De Groot (1967) that if a topological space T is first countable and Hausdorff but not compact, then its antispace T^* is compact but non-Hausdorff. We shall now show examples of some topological spaces which are first countable and Hausdorff and find out the topologies for their antispaces.

Let M denote Minkowski space of Special Relativity, i.e. a four-dimensional real vector space with the characteristic quadratic form Q:

$$M = \{x: x = (x_0, x_1, x_2, x_3), x_0, x_1, x_2, x_3 \text{ are reals}\}$$

$$Q(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

Let G be the smallest group of one-one mappings of M onto itself containing (i) the Lorentz group, i.e. all linear maps which keep the quadratic form Q invariant, (ii) translations of M (which also keep Q invariant) and (iii) dilatations of M (i.e. multiplication by scalars).

Let us have the following cones at x:

Light cone at x: $C^L(x) = \{y: Q(y-x) = 0\}$

Time cone at x: $C^T(x) = \{y: Q(y-x) > 0\} \cup \{x\}$

Space cone at x: $C^S(x) = \{y: Q(y-x) < 0\} \cup \{x\}$

A time-like line through x is defined as a line in the usual sense lying entirely in $C^T(x)$. Similarly, a light-like line through x is a line in the usual sense lying entirely in $C^L(x)$, and a space-like hyperplane through x is a hyperplane in the usual sense lying entirely in $C^S(x)$.

With these definitions, the t-topology and s-topology for M are defined as follows:

Let $N_\epsilon^t(x) = N_\epsilon^E(x) \cap C^T(x)$, where $N_\epsilon^E(x)$ denotes an Euclidean neighbourhood of x, i.e. $N_\epsilon^E(x) = \{y: d(x,y) < \epsilon\}$, where $d(x,y) = \{\sum_{i=0}^3 (x_i - y_i)^2\}^{1/2}$. The t-topology on M is defined by specifying a countable local base $\mathcal{N}(x)$ of neighbourhoods at each point x of M as follows:

$\mathcal{N}(x) = \{N_\epsilon^t(x): \epsilon > 0, \epsilon \text{ rational}\}$. Let (M, t) denote the space M equippped with the t-topology. Similarly, let $N_\epsilon^s(x) = N_\epsilon^E(x) \cap C^S(x)$. The s-topology on M is defined by specifying a countable local base $\mathcal{N}'(x)$ of neighbourhoods at each point x of M as follows:

$\mathcal{N}'(x) = \{N_\epsilon^s(x): \epsilon > 0, \epsilon \text{ rational}\}$. Let M equipped with the s-topology be denoted by (M, s).

For more detailed information about Minkowski space and its topologies and their homeomorphism groups, see Zeeman (1967) and Nanda (1971, 1972).

It is very easy to check that the t-topology and the s-topology are both finer than the Euclidean topology E on M and hence Hausdorff. Both of them are, by definition, first countable. The former induces the one-dimensional Euclidean topology on every time-like line and the discrete topology on every light-like line and space-like hyperplane. The latter induces the three-dimensional Euclidean topology on every space-like hyperplane, and the discrete topology on every time-like and light-like lines. Moreover, the compact sets in the t-topology are essentially of the form of closed and bounded subsets of time-like lines and the compact sets in s-topology are essentially three-dimensional closed balls contained in space-like hyperplanes. When we consider their antispaces (M, t^*) and (M, s^*), those two types of compact sets become closed sets in (M, t^*) and (M, s^*) respectively. Thus an open set about a point x in (M, t^*) is the complement of a closed and bounded interval on a time-like line not containing x and is therefore unbounded. Similarly, an open set about x in (M, s^*) is the complement of a closed three-dimensional ball on a space-like hyperplane not containing x and is therefore unbounded. Therefore, the open sets of such antispaces (M, t^*), (M, s^*) of (M, t) and (M, s) are of infinite diameter.

Most of the topologies on M given by Zeeman (1967) and Nanda (1971, 1972) have G as their homeomorphism group and are, moreover, first countable and Hausdorff. The antispaces of such topological spaces are, therefore, compact but non-Hausdorff and have the same homeomorphism group G according to Theorem 1. Moreover, the open sets of such antispaces are of infinite diameter. Thus, these antispaces provide us with a rich variety of compact topologies on M having G as their homeomorphism group. In fact, it can also be established that for any arbitrary non-Hausdorff topology on Minkowski space having G as the homeomorphism group, the open sets are necessarily unbounded.

Theorem 2. Let T be any arbitrary non-Hausdorff topology on Minkowski space having G as its homeomorphism group, then the open sets in (M, T) are of infinite diameter.

Proof. To show that any arbitrary open set about a point x of M is of infinite diameter, we need only to show that any arbitrary open set about the origin is of infinite diameter, because the translation maps are homeomorphisms.

Suppose to the contrary that there exists an open set O about the origin which is of finite diameter k. Since T is non-Hausdorff, there exists at least one pair of distinct points x, y of M such that if A and B are any two open sets about x and y respectively, then $A \cap B \neq \emptyset$. Let the distance between x and y be $3r$, i.e. $d(x, y) = 3r$, where d denotes the Euclidean distance function. Let $O' = (r/k)O$. Since the dilatation map is a homeomorphism, O' is again an open set about the origin and the diameter of O' = diameter $(r/k \,.\, O) = r/k \,.\,$ diameter of $O = r$. Let $O'_x = O' + x$ and $O'_y = O' + y$, i.e. O'_x and O'_y are translates of O' and are therefore open sets about x and y respectively. Moreover, each of them is of diameter r. By our assumption, $O'_x \cap O'_y \neq \emptyset$. Let $z \in O'_x \cap O'_y$. Since x and z are in O'_x if follows that $d(x, z) \leqslant r$ and similarly $d(y, z) \leqslant r$. Therefore, by triangle inequality, $3r = d(x, y) \leqslant d(x, z) + d(z, y) \leqslant 2r$, thus giving a contradition. The theorem is therefore proved.

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